

A PSEUDOSPECTRAL APPROXIMATION TO THE FUNDAMENTAL MATRIX OF A LINEAR DELAY DIFFERENTIAL EQUATION WITH PERIODIC COEFFICIENTS

DAVID E. GILSINN

Mathematical and Computational Sciences Division
National Institute of Standards and Technology
100 Bureau Drive, Stop 8910
Gaithersburg, MD 20899-8910 ¹

ABSTRACT. The monodromy operator of a linear delay differential equation with periodic coefficients is formulated as an integral operator. The kernel of this operator includes a factor formed from the fundamental solution of the linear delay differential equation. Although the properties of the fundamental solutions are known, in general there is no closed form for the fundamental solution. This paper describes a collocation procedure to approximate the fundamental solution before the integral operator is discretized.

1. Introduction

Delay differential equations have occurred in many fields from biology (MacDonald [27]) to population dynamics (Kuang [24]) to machine tool dynamics (Balchandran [2], Zhao and Balachandran [44], Nayfeh et al. [28]). The study of machine tool dynamics has led to many problems involving delay differential equations. For example, in turning operations a cutting tool passes over a workpiece many times successively. The forces on the tool depend on chip thickness which is dependent on the tool's current position and its position one previous revolution of the workpiece, thus introducing a delay effect. Any irregularities in a previous cut produced by the tool can affect the current cut. The delay effect of the irregularities can introduce self-sustained oscillations of the tool against the workpiece, called *regenerative chatter*. This phenomenon has been studied by Tlustý and Placek [37] and Tobias [38] as early as the 1960's. Mathematically, chatter is a stable limit cycle of the delay differential equation that models the particular machining process. To a machinist it represents undesirable motions that can damage a good surface finish. Therefore, being able to determine the nature of the stability of periodic solutions to delay differential equations is crucial to determining the quality of the workpiece surface finish. It will be seen that the nature of the stability is determined by the eigenvalues of a certain integral operator, called the monodromy operator, associated with the delay differential equation.

Many of the models of machining operations fall into the class of autonomous delay differential equations of the form

$$(1) \quad \dot{x} = X(x(t), x(t-h)),$$

where $x, X \in \mathbb{R}^n$, $h > 0$. A main concern is the question of stability of periodic solutions of (1). See, for example, Butcher et al. [10], Engelborghs et al. [12], Luzyanina and Engelborghs [25], and Luzyanina and Roose [26].

Since the period $T = 2\pi/\omega$ of a periodic solution for (1) is unknown we can normalize the period to $[0, 2\pi]$ by introducing the substitution of th/ω for t and rewriting (1) in the form

$$(2) \quad \omega \dot{x} = hX(x(t), x(t-\omega)).$$

Since h is simply a rescaling of the system we will drop it for the rest of the paper

The analysis of the stability of a periodic solution for (1) usually involves the following considerations. Let $p(t)$, $p \in \mathbb{R}^n$, be a periodic function of some period $T > 0$ that may or may not be an exact periodic solution of (1). This function may, for example, have been developed by a Galerkin method or harmonic

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balance. In this paper we will only consider the case of $T > \omega$. Then, the linear variational equation about the periodic function $p(t)$ can be written

$$(3) \quad \dot{z}(t) = A(t)z(t) + B(t)z(t - \omega),$$

where

$$(4) \quad \begin{aligned} A(t) &= X_1(p(t), p(t - \omega)), \\ B(t) &= X_2(p(t), p(t - \omega)). \end{aligned}$$

The subscripts represent the partial derivatives with respect to the first and second variables, respectively. Since $p(t)$ is periodic with period $T > 0$, $A(t)$ and $B(t)$ are clearly periodic, with the same period. Let $\mathcal{L} = \max(\|A(t)\|_\infty, \|B(t)\|_\infty)$, where $\|\cdot\|$ will be used to represent a matrix norm.

Let C_ω denote the space of continuous functions from $[-\omega, 0]$ to \mathbb{R}^n , with norm in C_ω given by $|\phi| = \max|\phi(s)|$ for $-\omega \leq s \leq 0$. C_ω is a Banach space with respect to this norm.

We define the period map $U : C_\omega \rightarrow C_\omega$ with respect to (3) by

$$(5) \quad (U\phi)(s) = z(s + T),$$

where $z(s)$ is a solution of (3) satisfying $z(s) = \phi(s)$ for $s \in [-\omega, 0]$. For $T > \omega$, U is a compact operator on C_ω , whose spectrum is at most countable with zero as the only possible limit point (Halanay [17]). Halanay [17] has also shown that the period operator U , also called the *monodromy operator*, can be represented as

$$(6) \quad (U\phi)(s) = Z(s + T, 0)\phi(0) + \int_{-\omega}^0 Z(s + T, \alpha + \omega)B(\alpha + \omega)\phi(\alpha) d\alpha,$$

where $Z(s, \alpha)$ is the fundamental solution of (3) which satisfies (3) for $s > \alpha$, $Z(\alpha, \alpha) = I_n$, the $n \times n$ identity matrix, and $Z(s, \alpha) = 0$ for $s < \alpha$. A finite monodromy matrix is obtained by discretizing (6). The specific monodromy matrix used depends on the choice of discretization method used.

The nature of the stability of the approximate periodic solution $p(t)$ to (3) depends on the eigenvalues of the period map (5). These eigenvalues are also referred to as *characteristic multipliers*, since

$$(7) \quad z(s + T) = (U\phi)(s) = \lambda\phi(s) = \lambda z(s)$$

for some λ and some $z(s) = \phi(s)$ not identically zero for $s \in [-\omega, 0]$. $\phi(s)$ in this case will be an eigenfunction of U . Along with its relation to stability the λ can also be thought about as a measure of how close $z(s)$ is to periodicity with period $T > 0$. If $\lambda = 1$, then $z(s)$ is a periodic solution with period $T > 0$. Although the term characteristic multiplier and eigenvalue of U are sometimes used interchangeably, we will maintain the usage of the term eigenvalue when referring to operators.

The computation of the eigenvalues for (6) generally involves some form of approximation. In this paper we will use two levels of approximation. The first level is the discretization of the integral and the next level is the approximation of the fundamental solution $Z(s, \alpha)$. Only in very rare cases is the fundamental solution exactly computable. The discretization of the monodromy operator (6) produces a matrix that is referred to as a discrete monodromy matrix. However, for simplicity, when the context is clear, it is referred to as a *monodromy matrix*. It, of course, depends on the quadrature method used. The main aim of this paper will be to estimate the fundamental solution $Z(s, \alpha)$ and compute the eigenvalues of (6).

In Section 2 we will present a Galerkin approximation approach to computing an approximate periodic solution to (2). In Section 3 we will develop a pseudo-spectral approximation to the fundamental matrix and prove an error analysis result for the approximation. In Section 4 we will discretize the monodromy operator and form an eigenvalue problem to approximate the the eigenvalues of the discretized operators. Finally, in Section 5 we compute the characteristic multipliers for the Van der Pol oscillator with delay.

2. Approximating a Solution and Frequency

In this section we present a Galerkin method for approximating a solution for (2). This is used to form the variational equation for which the fundamental matrix is computed in Section 3.

An approximate solution and frequency for (2) can be developed by assuming a finite trigonometric polynomial of the form

$$(8) \quad \hat{x}_m = a_2 \cos t + \sum_{n=2}^m [a_{2n} \cos nt + a_{2n-1} \sin nt]$$

where the $\sin t$ term has been dropped so that we can estimate $a_1 = \hat{\omega}$, the frequency. Some computational experience has suggested that dropping a low order harmonic term provides a smaller residual estimate for (15).

Note that we have centered the approximate solution about the origin, since we assumed $X(0, 0) = 0$. If we set $\bar{\mathbf{a}} = (a_1, a_2, \dots, a_{2m})$, and

$$(9) \quad E_m(t, \bar{\mathbf{a}}) = a_1 \dot{\hat{x}}_m(t) - X(\hat{x}_m(t), \hat{x}_m(t - a_1))$$

then for a sufficiently fine mesh, specified by $\{t_i : i = 1, 2, \dots, 2N\}$, in $[0, 2\pi]$, where

$$(10) \quad t_i = \frac{2i-1}{2N}\pi,$$

the determining equations for $\bar{\mathbf{a}}$ can be written as (see Urabe and Reiter [41])

$$\begin{aligned}
 F_1(\bar{\mathbf{a}}) &= \frac{1}{N} \sum_{i=1}^{2N} E_m(t_i, \bar{\mathbf{a}}) \sin t_i = 0 \\
 F_2(\bar{\mathbf{a}}) &= \frac{1}{N} \sum_{i=1}^{2N} E_m(t_i, \bar{\mathbf{a}}) \cos t_i = 0 \\
 F_{2n-1}(\bar{\mathbf{a}}) &= \frac{1}{N} \sum_{i=1}^{2N} E_m(t_i, \bar{\mathbf{a}}) \sin nt_i = 0 \\
 F_{2n}(\bar{\mathbf{a}}) &= \frac{1}{N} \sum_{i=1}^{2N} E_m(t_i, \bar{\mathbf{a}}) \cos nt_i = 0
 \end{aligned}
 \tag{11}$$

for $n = 2, \dots, m$.

These equations give $2m$ equations in $2m$ unknowns. Standard numerical solvers, using, for example, Newton's method, for nonlinear equations can be used to solve for $\bar{\mathbf{a}}$. The number of harmonics, m , and the quadrature index, N , can be selected independently. Gilsinn [14] presents a vectorized algorithm for solving for \bar{a} .

3. Approximating the Fundamental Matrices by Pseudo-spectral Collocation

Numerically approximating a solution to a delay differential equation has been studied by many authors. See, for example, Paul [30], Shampine and Thompson [33], Willé and Baker [43]. For a spectral method for solving delay differential equations with constant coefficients see Ito et al. [21]. In this paper we will seek a collocation representation for the fundamental solution. There have been some studies in which representational solutions to delay differential equations have been sought. In particular, Engelborghs et al. [12] studied collocation methods for computing periodic solutions for delay differential equations. There have been studies of spline approximations by Banks and Kappel [4] and Kemper [23]. For a symbolic method for computing fundamental solutions for time-periodic ordinary differential equations see Sinha and Butcher [34].

3.1. The Method of Steps. Let $C_\omega(a)$ denote the space of continuous functions from $[a - \omega, a]$ to \mathbb{R}^n , with norm in $C_\omega(a)$ given by $|\phi| = \max |\phi(s)|$ for $a - \omega \leq s \leq a$. Note that $C_\omega = C_\omega(0)$. We wish to solve the linear delay differential equation

$$\dot{z}(t) = A(t)z(t) + B(t)z(t - \omega), \tag{12}$$

where $A(t)$ and $B(t)$, given by (4), are $n \times n$ matrices of continuous functions, periodic with period $T > 0$ over the interval $[a, b]$ with b finite. The initial condition is given by $z(t) = \phi(t)$ on $[a - \omega, a]$, $z(a) = z_0$.

The object of the *method of steps* is to reduce the problem of directly solving the delay equation (12) to solving a finite sequence of ordinary differential equations. This method has been used for many years in delay differential equations (see e. g. Bellman and Cooke [8]).

In the present context we begin by first finding the smallest positive integer q such that $a + q\omega \geq b$. The integer q depends on ω , but ω is fixed for a given problem. We now consider the finite set of intervals $[a, a + \omega]$, $[a + \omega, a + 2\omega]$, \dots , $[a + (q - 1)\omega, a + q\omega]$, where the point b falls within the last interval. If b is an exact multiple, then $a + q\omega = b$.

At the first step,

$$\dot{z}_1(t) = A(t)z_1(t) + B(t)z_1(t - \omega), \tag{13}$$

where $z_1(t - \omega) = \phi(s)$ for some initial function $\phi \in C_\omega(a)$ and $s = t - \omega$. Thus the initial problem becomes an ordinary differential equation. Then, on $[a + \omega, a + 2\omega]$ we solve

$$\dot{z}_2(t) = A(t)z_2(t) + B(t)z_2(t - \omega), \tag{14}$$

where $z_2(a + \omega) = z_1(a + \omega)$, $z_2(t - \omega) = z_1(s)$ for $s \in [a, a + \omega]$, $s = t - \omega$. Again, we solve (14) as an ordinary differential equation. The process is continued so that on $[a + (i - 1)\omega, a + i\omega]$, for $i = 1, 2, \dots, q$,

$$\dot{z}_i(t) = A(t)z_i(t) + B(t)z_i(t - \omega), \tag{15}$$

with $z_i(a + (i - 1)\omega) = z_{i-1}(a + (i - 1)\omega)$. We then define $z(t)$ on $[a, b]$ as the concatenation of $z_i(t)$ for $t \in [a + (i - 1)\omega, a + i\omega]$ and $i = 1, 2, \dots, q$.

3.2. A Sequence of Differential Equations. Since we wish to use a Chebyshev collocation method to solve each of the differential equations (15) for $i = 1, 2, \dots, q$, we will normalize each of the intervals $[a + (i - 1)\omega, a + i\omega]$ to $[-1, 1]$ as we step through the finite sequence of differential equations (15).

The unique transformation between $[a + (i - 1)\omega, a + i\omega]$, for $i = 1, 2, \dots, q$, and $[-1, 1]$ is given as follows. For each $t \in [a + (i - 1)\omega, a + i\omega]$ for $i = 1, 2, \dots, q$, there is a unique $\eta \in [-1, 1]$ given by

$$\eta = \frac{2}{\omega}t - \frac{(2a + (2i - 1)\omega)}{\omega}. \tag{16}$$

For $\eta \in [-1, 1]$ we have the unique $t \in [a + (i-1)\omega, a + i\omega]$ given by

$$(17) \quad t = \frac{\omega}{2}\eta + \frac{(2a + (2i-1)\omega)}{2}.$$

We note that the points $t \in [a + (i-1)\omega, a + i\omega]$ and $t - \omega \in [a + (i-2)\omega, a + (i-1)\omega]$ are translated to the same $\eta \in [-1, 1]$. This is clear from

$$(18) \quad \frac{2}{\omega}(t - \omega) - \frac{(2a + (2i-3)\omega)}{\omega} = \frac{2}{\omega}t - \frac{(2a + (2i-1)\omega)}{\omega}.$$

We can now shift the solving of the sequence of delay problems

$$(19) \quad \dot{z}_i(t) = A(t)z_i(t) + B(t)z_i(t - \omega),$$

for $t \in [a + (i-1)\omega, a + i\omega]$ and $i = 1, 2, \dots, q$, into solving a sequence of ordinary differential equations

$$(20) \quad u'_i(\eta) = \frac{\omega}{2}\hat{A}_i(\eta)u_i(\eta) + \frac{\omega}{2}\hat{B}_i(\eta)u_{i-1}(\eta),$$

where, for $t \in [a + (i-1)\omega, a + i\omega]$,

$$(21) \quad \begin{aligned} u_i(-1) &= u_{i-1}(1), \\ u_i(\eta) &= z_i(t), \\ \hat{A}_i(\eta) &= A(t), \\ \hat{B}_i(\eta) &= B(t), \\ u_{i-1}(\eta) &= z_i(t - \omega). \end{aligned}$$

The initial function is

$$(22) \quad u_0(\eta) = z_1(t - \omega) = \phi(t - \omega), \quad t - \omega \in [a - \omega, a].$$

Let the columns of the identity matrix I_n be written as $e_j = (0, \dots, 1, \dots, 0)^T$, where one is the j -th element and all others are zero. We can now approximate the fundamental solution for (3) on $[a, b]$ by first solving n sequences of q differential equations (20) subject to

$$(23) \quad \begin{aligned} u_i(-1) &= u_{i-1}(1) \\ u_0(\eta) &= 0, \quad \eta \in [-1, 1] \\ u_1(-1) &= e_j \end{aligned}$$

where $j = 1, \dots, n$. We can then transform back to the t domain. In the next section we show how the Lagrange polynomials can be used to develop a collocation solution to each differential equation in the sequence (20).

3.3. A Pseudo-spectral Collocation Algorithm. In this section we follow the pseudo-spectral method used by Bueler [9]. Although an analysis of the stability of pseudo-spectral methods for partial differential equations has been given by Gottlieb [15], we will develop a separate error analysis result in the next section for the pseudo-spectral method described here.

We first define a sequence of projection operators. Let N be a positive integer. Let P_N be the projection operator that associates a continuous function f defined on $[-1, 1]$ with the unique N -th degree Lagrange polynomial interpolating through the $N + 1$ Chebyshev extreme points

$$(24) \quad \eta_k = \cos\left(\frac{k\pi}{N}\right), \quad k = 0, 1, \dots, N.$$

If the Lagrange interpolation polynomials at these points are given by

$$(25) \quad l_j(\eta) = \prod_{\substack{k=0 \\ k \neq j}}^N \left(\frac{\eta - \eta_k}{\eta_j - \eta_k} \right).$$

for $j = 1, 2, \dots, N$ and $l_j(\eta_k) = \delta_{jk}$, where $\delta_{jj} = 1$, $\delta_{jk} = 0$, $j \neq k$, then

$$(26) \quad (P_N f)(\eta) = \sum_{k=0}^N f(\eta_k) l_j(\eta)$$

We have that $|P_N| \leq \mathcal{P}$ for some $\mathcal{P} > 0$ by the Banach-Steinhaus Theorem.

For $\eta \in [-1, 1]$ we set

$$(27) \quad \hat{u}_i(\eta) = \sum_{j=0}^N w_j^{(i)} l_j(\eta),$$

where the subscript i indicates that η is associated with the unique $t \in [a + (i-1)\omega, a + i\omega]$, $w_j^{(i)}$ is an n -vector to be determined, and the hat is intended to indicate a solution to the problem

$$(28) \quad \hat{u}'_i(\eta) = P_N \left(\frac{h}{2} \hat{A}_i \hat{u}_i(\eta) + \frac{h}{2} \hat{B}_i \hat{u}_{i-1}(\eta) \right),$$

subject to the collocation condition described below.

We also need to form

$$(29) \quad \widehat{u}'_i(\eta) = \sum_{j=0}^N w_j^{(i)} l_j'(\eta).$$

At the Chebyshev points we will designate

$$(30) \quad D_{kj} = l_j'(\eta_k).$$

The values for these derivatives are given in Gottlieb and Turkel [16] or Trefethen [39] but we state the values for D_{kj} here for completeness.

$$(31) \quad \begin{aligned} D_{00} &= \frac{2N^2 + 1}{6}, \\ D_{NN} &= -D_{00}, \\ D_{jj} &= \frac{-\eta_j}{2(1 - \eta_j^2)}, \quad j = 1, 2, \dots, N-1 \\ D_{ij} &= \frac{c_i(-1)^{i+j}}{c_j(\eta_i - \eta_j)}, \quad i \neq j, i, j = 0, \dots, N, \end{aligned}$$

where

$$(32) \quad c_i = \begin{cases} 2, & i = 0 \text{ or } N; \\ 1, & \text{otherwise.} \end{cases}$$

One of the principal reasons for selecting the Chebyshev points (24) is that the pseudo-spectral differentiation matrix (31) is known exactly. Some further advantages are discussed in Salzer [32].

For notation, let

$$(33) \quad \begin{aligned} \widehat{u}_i(\eta) &= (u_{i1}(\eta), \dots, u_{in}(\eta))^T, \\ \widehat{A}_i(\eta) &= [\widehat{A}_{pq}^{(i)}(\eta)]_{p,q=1,\dots,n}, \\ \widehat{B}_i(\eta) &= [\widehat{B}_{pq}^{(i)}(\eta)]_{p,q=1,\dots,n}. \end{aligned}$$

We then write the collocation polynomial elements of $\widehat{u}_i(\eta)$ as $\widehat{u}_{ir}(\eta)$, $r = 1, \dots, n$, where

$$(34) \quad \widehat{u}_{ir}(\eta) = \sum_{k=0}^N w_{rk}^{(i)} l_k(\eta),$$

at the Chebyshev points (24) to get

$$(35) \quad \begin{aligned} \widehat{u}_{ir}(\eta_j) &= w_{rj}^{(i)}, \\ \widehat{u}'_{ir}(\eta_j) &= \sum_{k=0}^N w_{rk}^{(i)} D_{jk}, \\ \widehat{u}_{i-1,r}(\eta_j) &= w_{rj}^{(i-1)}. \end{aligned}$$

The initial conditions for the sequence of differential equation are

$$(36) \quad \widehat{u}_{ir}(\eta_N) = \widehat{u}_{i-1,r}(\eta_0),$$

or

$$(37) \quad w_{rN}^{(i)} = w_{r0}^{(i-1)},$$

for $r = 1, \dots, n$.

The discretized differential equations are then given by

$$(38) \quad \begin{pmatrix} \sum_{1k}^{(i)} w_{1k}^{(i)} D_{jk} \\ \vdots \\ \sum_{rk}^{(i)} w_{rk}^{(i)} D_{jk} \\ \vdots \\ \sum_{nk}^{(i)} w_{nk}^{(i)} D_{jk} \end{pmatrix} = \frac{h}{2} \begin{bmatrix} \hat{A}_{11}^{(i)}(\eta_j) & \cdots & \hat{A}_{1n}^{(i)}(\eta_j) \\ \vdots & \cdots & \vdots \\ \hat{A}_{r1}^{(i)}(\eta_j) & \cdots & \hat{A}_{rn}^{(i)}(\eta_j) \\ \vdots & \cdots & \vdots \\ \hat{A}_{n1}^{(i)}(\eta_j) & \cdots & \hat{A}_{nn}^{(i)}(\eta_j) \end{bmatrix} \begin{pmatrix} w_{1j}^{(i)} \\ \vdots \\ w_{rj}^{(i)} \\ \vdots \\ w_{nj}^{(i)} \end{pmatrix} + \frac{h}{2} \begin{bmatrix} \hat{B}_{11}^{(i)}(\eta_j) & \cdots & \hat{B}_{1n}^{(i)}(\eta_j) \\ \vdots & \cdots & \vdots \\ \hat{B}_{r1}^{(i)}(\eta_j) & \cdots & \hat{B}_{rn}^{(i)}(\eta_j) \\ \vdots & \cdots & \vdots \\ \hat{B}_{n1}^{(i)}(\eta_j) & \cdots & \hat{B}_{nn}^{(i)}(\eta_j) \end{bmatrix} \begin{pmatrix} w_{1j}^{(i-1)} \\ \vdots \\ w_{rj}^{(i-1)} \\ \vdots \\ w_{nj}^{(i-1)} \end{pmatrix}$$

for $j = 0, 1, \dots, N-1$. These provide nN equations in $n(N-1)$ unknowns. The other n equations come from the initial conditions. We can write this system in a more compact form by first defining the following vectors

$$(39) \quad \begin{aligned} w_i &= \left(w_{10}^{(i)} \cdots w_{1N}^{(i)} w_{20}^{(i)} \cdots w_{2N}^{(i)} \cdots w_{n0}^{(i)} \cdots w_{nN}^{(i)} \right)^T, \\ w_{i-1} &= \left(w_{10}^{(i-1)} \cdots w_{1N}^{(i-1)} w_{20}^{(i-1)} \cdots w_{2N}^{(i-1)} \cdots w_{n0}^{(i-1)} \cdots w_{nN}^{(i-1)} \right)^T. \end{aligned}$$

We also define matrices \hat{A}_i and \hat{B}_i .

$$(40) \quad \hat{A}_i = \begin{bmatrix} \hat{A}_{11}^{(i)}(\eta_0) & 0 & \cdots & 0 & \cdots & 0 & \hat{A}_{1n}^{(i)}(\eta_0) & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \cdots & 0 & 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \hat{A}_{11}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 & 0 & \hat{A}_{1n}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hat{A}_{n1}^{(i)}(\eta_0) & 0 & \cdots & 0 & \cdots & 0 & \hat{A}_{nn}^{(i)}(\eta_0) & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \cdots & 0 & 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \hat{A}_{n1}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 & 0 & \hat{A}_{nn}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

\hat{B}_i is structured in a similar manner except every $(N+1)$ th row includes an element $2/h$ to take care of the initial condition by canceling the $h/2$ in (38). Thus

$$(41) \quad \hat{B}_i = \begin{bmatrix} \hat{B}_{11}^{(i)}(\eta_0) & 0 & \cdots & 0 & \cdots & 0 & \hat{B}_{1n}^{(i)}(\eta_0) & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \cdots & 0 & 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \hat{B}_{11}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 & 0 & \hat{B}_{1n}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 \\ \frac{2}{h} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hat{B}_{n1}^{(i)}(\eta_0) & 0 & \cdots & 0 & \cdots & 0 & \hat{B}_{nn}^{(i)}(\eta_0) & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \cdots & 0 & 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \hat{B}_{n1}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 & 0 & \hat{B}_{nn}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \frac{2}{h} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then we can write the sequence of differential equations in vector form as

$$(42) \quad \hat{D}w_i = \frac{h}{2}\hat{A}_i w_i + \frac{h}{2}\hat{B}_i w_{i-1},$$

where \hat{D} is the Kronecker product, $\hat{D} = D \otimes I_n$, and each D is given by

$$(43) \quad D = \begin{bmatrix} D_{00} & \cdots & D_{0N} \\ \vdots & \ddots & \vdots \\ D_{N-1,0} & \cdots & D_{N-1,N} \\ 0 & \cdots & 1 \end{bmatrix}.$$

The Kronecker product produces n blocks of D arrays down the diagonal. The unit in the lower right introduces the n initial condition, $w_{rN}^{(i)}$, $r = 1, \dots, n$.

The linear equation (42) can be solved for w_i by setting

$$(44) \quad M_i = \left(\hat{D} - \frac{h}{2}\hat{A}_i \right)^{-1} \frac{h}{2}\hat{B}_i$$

and

$$(45) \quad w_i = M_i w_{i-1}$$

for $i = 2, 3, \dots, q$. The ability to take the inverse here, for each i , must be determined numerically. There has been some study of the eigenvalues of the matrix D done by Trefethen and Trummer [40] as well as small perturbation of matrix eigenvalues by Kato [22], but there does not seem to be a definitive study of the large perturbations of the differentiation matrix, D .

To solve for w_1 , for the fundamental solution, we need the collocation solution of

$$(46) \quad \hat{u}'_1(\eta) = \frac{h}{2} \hat{A}_1(\eta) \hat{u}_1(\eta)$$

for $\eta \in [-1, 1]$ and

$$(47) \quad \hat{u}_1(-1) = e_j.$$

That is, we solve n problems at each step, one for each of the initial conditions e_j . For the moment we set the initial vector as

$$(48) \quad w_0 = (0 \cdots u_{01} 0 \cdots u_{02} 0 \cdots u_{0n})^T,$$

where u_{0r} , $r = 1, \dots, n$, is placed in each of the $(N+1)$ th elements and zero elsewhere. Then from the previous construction of \hat{D} and \hat{A}_1 we have

$$(49) \quad w_1 = \left(\hat{D} - \frac{h}{2} \hat{A}_1 \right)^{-1} w_0.$$

Now, given that we have computed

$$(50) \quad \hat{u}_{ir}(\eta) = \sum_{k=0}^N w_{rk}^{(i)} l_k(\eta)$$

for $\eta \in [-1, 1]$, for $r = 1, \dots, n$, we can compute the result for $t \in [a + (i-1)\omega, a + i\omega]$ by setting

$$(51) \quad \hat{z}_{ir}(t) = \hat{u}_{ir}(\eta)$$

for $r = 1, \dots, n$, where

$$(52) \quad \eta = \frac{2}{\omega} t - \frac{(2a + (2i-1)\omega)}{\omega},$$

or

$$(53) \quad \hat{z}_{ir}(t) = \sum_{k=0}^N w_{rk}^{(i)} l_k \left(\frac{2}{\omega} t - \frac{(2a + (2i-1)\omega)}{\omega} \right).$$

The initial condition is

$$(54) \quad \hat{u}_{ir}(\eta_N) = \hat{u}_{i-1,r}(\eta_0).$$

But on $[a + (i-1)\omega, a + i\omega]$, $\eta_N = -1$ corresponding to $t = a + (i-1)\omega$, and on $[a + (i-2)\omega, a + (i-1)\omega]$, $\eta_0 = 1$ corresponding to $t = a + (i-1)\omega$, so that

$$(55) \quad \hat{z}_{ir}(a + (i-1)\omega) = \hat{z}_{i-1,r}(a + (i-1)\omega).$$

The fundamental solution is formed as n column vectors. Each j -th column vector, $j = 1, \dots, n$, is formed as follows. The initial condition $e_j = (0, \dots, 1, \dots, 0)^T$ is selected. The r -th element, $r = 1, \dots, n$, in the column is formed by concatenating the functions $z_{ir}(t)$ for $i = 1, \dots, q$. The final matrix is then denoted by $\hat{Z}_N(t, a)$ where a is an initial point, not necessarily zero.

For further discussion of the numerical aspects of computing differentiation matrices see Baltensperger and Berrut [3], Bayliss et al. [5], Funaro [13], Solomonoff [35], and Welfert [42].

3.4. Error Estimates. In this section we develop an error estimate between the pseudospectral collocation of the fundamental solution of (12) and the exact fundamental solution of (12). In fact we are able to prove, using a method of Bellen [6], the next theorem, where \mathcal{L} and \mathcal{P} have been defined previously in Sections 1 and 3.3, respectively. Similar methods have been used by de Boor and Swartz [11], Russell and Shampine [31], and Hulme [20]. See also Bellen and Zennaro [7].

We suppose that $N+1$ Chebyshev extreme points (24) are given in $[-1, 1]$. Define $\Delta = \max |\eta_{i+1} - \eta_i|$. Then it is easy to show that $\Delta \leq \pi/N$.

Theorem 3.1. *Let $\mathcal{L}_1 = (h/2)\mathcal{P}\mathcal{L}$. Choose N sufficiently large so that $\Delta < 1/\mathcal{L}_1$. If $Z(t, a)$ is the exact fundamental solution of (12) for $t \in [a, b]$, where the right hand side of (12) is assumed to have bounded derivatives of sufficiently high order, and $\hat{Z}_N(t, a)$ is the approximate fundamental solution developed by the pseudo-spectral method of Section 3.3, then $\|Z - \hat{Z}_N\|_\infty = O(\Delta^{N+2})$.*

Proof. Since each of the q intervals in the method of steps is mapped to $[-1, 1]$ we will work with the sequence of differential equations (20). With this in mind we begin the error estimates at the first step, where the exact equation, in matrix form, is given by

$$(56) \quad V_1'(\eta) = \frac{h}{2} \hat{A}_1(\eta) V_1(\eta),$$

with the initial condition $V_1(-1) = I_n$. In this section the V will represent the matrix solutions of the differential equations. At this step we seek the matrix of Lagrange interpolation polynomials that collocate at the $N + 1$ Chebyshev extreme points in $[-1, 1]$ that satisfy

$$(57) \quad \hat{V}_1'(\eta) = \frac{h}{2} P_N \left(\hat{A}_1(\eta) \hat{V}_1(\eta) \right).$$

Since there exists a unique solution to (56) we can subtract (57) to form

$$(58) \quad V_1'(\eta) - \hat{V}_1'(\eta) = \frac{h}{2} \hat{A}_1(\eta) V_1(\eta) - \frac{h}{2} P_N \left(\hat{A}_1(\eta) \hat{V}_1(\eta) \right).$$

For simplicity we will re-index the Chebyshev points so that $\zeta_i = \eta_{N-i} = \cos((N-i)\pi/N)$ for $i = 0, 1, \dots, N$. We now add and subtract the projection of the right hand side of (56) to get

$$(59) \quad \begin{aligned} V_1(\eta) - \hat{V}_1(\eta) &= V_1(\zeta_i) - \hat{V}_1(\zeta_i) \\ &+ \frac{h}{2} \int_{\zeta_i}^{\eta} (I - P_N) \left(\hat{A}_1(s) V_1(s) \right) ds \\ &+ \frac{h}{2} \int_{\zeta_i}^{\eta} P_N \left(\hat{A}_1(s) (V_1(s) - \hat{V}_1(s)) \right) ds, \end{aligned}$$

where I is the identity operator and we also use the fact that P_N is linear.

We need to compute some bounds at this point. In particular, from Lagrange interpolation theory we have

$$(60) \quad \left\| (I - P_N) \left(\hat{A}_1(s) V_1(s) \right) \right\|_{\infty} \leq \mathcal{K}_0 \Delta^{N+1},$$

for some \mathcal{K}_0 . Although \mathcal{K}_0 in general depends on N , we assume here that the derivatives of the right hand side of (56) are bounded. We also have $\left\| \hat{A}_i(s) \right\|_{\infty} \leq \mathcal{L}$ and $\left\| \hat{B}_i(s) \right\|_{\infty} \leq \mathcal{L}$ for $i = 1, 2, \dots, q$.

Now let

$$(61) \quad e_i = \max_{-1 \leq \eta \leq \zeta_i} \left\| V_1(\eta) - \hat{V}_1(\eta) \right\|_{\infty}.$$

Then,

$$(62) \quad \begin{aligned} e_{i+1} &\leq e_i + \frac{h}{2} \int_{\zeta_i}^{\zeta_{i+1}} \left\| (I - P_N) \left(\hat{A}_1(s) V_1(s) \right) \right\|_{\infty} ds \\ &+ \frac{h}{2} \int_{\zeta_i}^{\zeta_{i+1}} \left\| P_N \left(\hat{A}_1(s) (V_1(s) - \hat{V}_1(s)) \right) \right\|_{\infty} ds, \end{aligned}$$

or, letting $\Delta = \max(\zeta_{i+1} - \zeta_i)$,

$$(63) \quad e_{i+1} \leq e_i + \frac{h}{2} \mathcal{K}_0 \Delta^{N+2} + \frac{h}{2} \mathcal{P} \mathcal{L} \Delta e_{i+1}.$$

If we set $\mathcal{K}_1 = \frac{h}{2} \mathcal{K}_0$, then (63) becomes

$$(64) \quad e_{i+1} \leq e_i + \mathcal{K}_1 \Delta^{N+2} + \mathcal{L}_1 \Delta e_{i+1}.$$

For Δ sufficiently small,

$$(65) \quad e_{i+1} \leq \left(\frac{1}{1 - \mathcal{L}_1 \Delta} \right) e_i + \left(\frac{\mathcal{K}_1}{1 - \mathcal{L}_1 \Delta} \right) \Delta^{N+2},$$

for $i = 0, 1, \dots, N$. To simplify a little, let

$$(66) \quad \alpha = \frac{1}{1 - \mathcal{L}_1 \Delta}, \quad \beta = \left(\frac{\mathcal{K}_1}{1 - \mathcal{L}_1 \Delta} \right) \Delta^{N+2},$$

then, by a simple geometric series argument,

$$(67) \quad e_N \leq \alpha^N e_0 + \left(\frac{\alpha^N - 1}{\alpha - 1} \right) \beta.$$

For the first step $e_0 = 0$, so that

$$(68) \quad e_N \leq \left(\frac{\alpha^N - 1}{\alpha - 1} \right) \beta = \left[\left(\frac{1}{1 - \mathcal{L}_1 \Delta} \right)^N - 1 \right] \frac{\mathcal{K}_1}{\mathcal{L}_1} \Delta^{N+1}.$$

Now, for $\Delta = \max(\zeta_{i+1} - \zeta_i) \leq \pi/N < 1/\mathcal{L}_1$, $(1/(1 - \mathcal{L}_1 \Delta))^N$ is a decreasing sequence, so that $(1/(1 - \mathcal{L}_1 \Delta))^N \leq (1/(1 - \mathcal{L}_1 \Delta))$. Therefore,

$$(69) \quad e_N \leq \left[\left(\frac{1}{1 - \mathcal{L}_1 \Delta} \right) - 1 \right] \frac{\mathcal{K}_1}{\mathcal{L}_1} \Delta^{N+1} = \left[\frac{\mathcal{K}_1}{1 - \mathcal{L}_1 \Delta} \right] \Delta^{N+2} \leq 2\mathcal{K}_1 \Delta^{N+2},$$

for Δ sufficiently small. Therefore

$$(70) \quad \max_{-1 \leq \eta \leq 1} \|V_1(\eta) - \widehat{V}_1(\eta)\|_\infty = O(\Delta^{N+2}).$$

We proceed inductively and assume that for the j -th step, $j = 1, 2, \dots, q-1$

$$(71) \quad \max_{-1 \leq \eta \leq 1} \|V_j(\eta) - \widehat{V}_j(\eta)\|_\infty \leq \beta_j \Delta^{N+2},$$

for some $\beta_j > 0$. We then have at the $(j+1)$ -th step the exact problem

$$(72) \quad V'_{j+1}(\eta) = \frac{h}{2} \widehat{A}_{j+1}(\eta) V_{j+1}(\eta) + \frac{h}{2} \widehat{B}_{j+1}(\eta) V_j(\eta).$$

with the initial condition $V_{j+1}(-1) = V_j(1)$. Proceeding as in the first step we seek the collocated solution to

$$(73) \quad \widehat{V}'_{j+1}(\eta) = \frac{h}{2} P_N \left(\widehat{A}_{j+1}(\eta) \widehat{V}_{j+1}(\eta) + \widehat{B}_{j+1}(\eta) \widehat{V}_j(\eta) \right),$$

with the initial condition $\widehat{V}_{j+1}(-1) = \widehat{V}_j(1)$. Now, subtracting (73) from (72) we have on $[\zeta_i, \zeta_{i+1}]$

$$(74) \quad \begin{aligned} V_{j+1}(\eta) - \widehat{V}_{j+1}(\eta) &= V_{j+1}(\zeta_i) - \widehat{V}_{j+1}(\zeta_i) \\ &\quad + \frac{h}{2} \int_{\zeta_i}^{\eta} (I - P_N) \left(\widehat{A}_{j+1}(s) V_{j+1}(s) + \widehat{B}_{j+1}(s) V_j(s) \right) ds \\ &\quad + \frac{h}{2} \int_{\zeta_i}^{\eta} P_N \left(\widehat{A}_{j+1}(s) (V_{j+1}(s) - \widehat{V}_{j+1}(s)) \right) ds \\ &\quad + \frac{h}{2} \int_{\zeta_i}^{\eta} P_N \left(\widehat{B}_{j+1}(s) (V_j(s) - \widehat{V}_j(s)) \right) ds. \end{aligned}$$

As before, let

$$(75) \quad e_i = \max_{-1 \leq \eta \leq \zeta_i} \|V_{j+1}(s) - \widehat{V}_{j+1}(s)\|_\infty.$$

From Lagrange interpolation theory we have

$$(76) \quad \|(I - P_N) \left(\widehat{A}_{j+1}(s) V_{j+1}(s) + \widehat{B}_{j+1}(s) V_j(s) \right)\|_\infty \leq \mathcal{K}_j \Delta^{N+1},$$

for some $\mathcal{K}_j > 0$. From the definition of \mathcal{L} we have $\|\widehat{A}_{j+1}(s)\|_\infty \leq \mathcal{L}$ and $\|\widehat{B}_{j+1}(s)\|_\infty \leq \mathcal{L}$. Then, from the definitions of \mathcal{L} , \mathcal{P} , \mathcal{K}_j ,

$$(77) \quad e_{i+1} \leq e_i + \frac{h}{2} (\mathcal{K}_j + \mathcal{P} \mathcal{L} \Delta \beta_j) \Delta^{N+2} + \frac{h}{2} \mathcal{P} \mathcal{L} \Delta e_{i+1}.$$

As in the initial case let $\mathcal{K}_{j+1} = \frac{h}{2} (\mathcal{K}_j + \mathcal{P} \mathcal{L} \Delta \beta_j)$ and $\mathcal{L}_{j+1} = \mathcal{L}_1 = \frac{h}{2} \mathcal{P} \mathcal{L}$, then for sufficiently small Δ we have

$$(78) \quad e_{i+1} \leq \left(\frac{1}{1 - \mathcal{L}_{j+1} \Delta} \right) e_i + \left(\frac{\mathcal{K}_{j+1}}{1 - \mathcal{L}_{j+1} \Delta} \right) \Delta^{N+2}.$$

As before, let $\alpha = \frac{1}{1 - \mathcal{L}_{j+1} \Delta}$ and $\beta = \frac{\mathcal{K}_{j+1}}{1 - \mathcal{L}_{j+1} \Delta}$, then

$$(79) \quad e_N \leq \alpha^N e_0 + \left(\frac{\alpha^N - 1}{\alpha - 1} \right) \beta \Delta^{N+2}.$$

But from the inductive assumption (71) $e_0 \leq \beta_j \Delta^{N+2}$, so that

$$(80) \quad e_N \leq \alpha^N \beta_j \Delta^{N+2} + \left(\frac{\alpha^N - 1}{\alpha - 1} \right) \beta \Delta^{N+2}.$$

Again, for $\Delta < 1/\mathcal{L}_1$, α^N is decreasing so that $\alpha^N \leq \alpha$ and therefore

$$(81) \quad e_N \leq (\alpha \beta_j + \beta) \Delta^{N+2}.$$

Therefore, the induction shows that

$$(82) \quad \max_{1 \leq i \leq q} \|V_i - \widehat{V}_i\|_\infty = O(\Delta^{N+2}).$$

But, since $Z_i(t) = V_i(\eta)$ and $\widehat{Z}_i(t) = \widehat{V}_i(\eta)$ we have from (82) and the fact that the fundamental solution $Z(t, a)$ is the concatenation of $Z_i(t, a) = V_i(\eta)$ and the collocated solution $\widehat{Z}_N(t, a)$ is the concatenation of $\widehat{Z}_{Ni}(t, a) = \widehat{V}_i(\eta)$ for $i = 1, 2, \dots, q$, that

$$(83) \quad \|Z - \widehat{Z}_N\|_\infty = \max_{1 \leq i \leq q} \|Z_i - \widehat{Z}_{Ni}\|_\infty = \max_{1 \leq i \leq q} \|V_i - \widehat{V}_i\|_\infty = O(\Delta^{N+2}).$$

□

This theorem implies that $\widehat{Z}_N(t, v)$ converges uniformly to $Z(t, v)$ for $(t, v) \in [0, T] \times [0, T]$.

4. Forming the Monodromy Matrix Eigenvalue Problem

In Section 1 the the monodromy operator (6) involved knowledge of the fundamental matrix of (3). In Section 3 a collocation algorithm to approximate the fundamental matrix along with an error analysis of the collocation method was developed. In this section we will define an approximate monodromy operator and use it to form a matrix eigenvalue problem to estimate the monodromy operator eigenvalues.

4.1. Discretized Operators. To approximate the monodromy operator (6) we will require a quadrature rule that satisfies

$$(84) \quad \sum_{k=1}^{P+1} v_k f(s_k) \rightarrow \int_{-h}^0 f(s) ds$$

as $P \rightarrow \infty$ for each continuous function $f \in C_h$. The rule is satisfied if

$$(85) \quad \sum_{k=1}^{P+1} |v_k| \leq M,$$

for some $M > 0$ and $P = 1, 2, \dots$.

Let $-\omega = s_1 < s_2 < \dots < s_{P+1} = 0$, and define

$$(86) \quad (U_P \phi)(s) = Z(s+T, 0)\phi(0) + \sum_{k=1}^{P+1} v_k Z(s+T, s_k + \omega) B(s_k + \omega) \phi(s_k)$$

for $\phi \in C_\omega$. From the theory of delay differential equations (Hale and Lunel [19]) it is known that $Z(t, u)$ is a continuous function on $[0, T] \times [0, T]$ and thus also uniformly continuous there. Furthermore, $B(t)$ is continuous and periodic in $[0, T]$.

Since the fundamental matrix is seldom known in practice we approximated it in Section 3. In that case we developed a sequence, \widehat{Z}_N , of matrices that converged uniformly to Z (Theorem 3.1). Now define a double sequence of operators on C_ω by

$$(87) \quad (U_{PN} \phi)(s) = \widehat{Z}_N(s+T, 0)\phi(0) + \sum_{k=1}^{P+1} v_k \widehat{Z}_N(s+T, s_k + \omega) B(s_k + \omega) \phi(s_k)$$

for $\phi \in C_\omega$, $P = 1, 2, \dots$, $N = 1, 2, \dots$.

4.2. The Matrix Eigenvalue Problem. From the discussion in Section 1, the stability of the approximate periodic solution $p(t)$ of (1) depends on the eigenvalues of the operator (6). In this section we will consider the discretized form of (6) given by (87) and point out some computational simplifications involved with constructing the approximate eigenvalue problem.

We discretize the interval $[-\omega, 0]$ into P intervals by

$$(88) \quad -\omega = s_1 < s_2 < \dots < s_{P+1} = 0.$$

Then, from (6), for each $s_i \in [-\omega, 0]$,

$$(89) \quad (U \phi)(s_i) = Z(s_i + T, 0)\phi(0) + \sum_{j=1}^{P+1} v_j Z(s_i + T, s_j + \omega) B(s_j + \omega) \phi(s_j)$$

Since $s_{P+1} = 0$, (89) can be rewritten as

$$(90) \quad \begin{aligned} (U \phi)(s_i) &= \sum_{j=1}^P v_j Z(s_i + T, s_j + \omega) B(s_j + \omega) \phi(s_j) \\ &+ (Z(s_i + T, 0) + v_{P+1} Z(s_i + T, \omega) B(\omega)) \phi(s_{P+1}), \end{aligned}$$

where $Z(t, \alpha)$, for $0 \leq \alpha \leq t \leq T$, is the fundamental solution of (3). Equation (90) can be put in matrix form

$$(91) \quad \begin{pmatrix} (U^{(P)} \phi)(s_1) \\ \vdots \\ (U^{(P)} \phi)(s_i) \\ \vdots \\ (U^{(P)} \phi)(s_{P+1}) \end{pmatrix} = \begin{bmatrix} U_{1,1}^{(P)} & \cdots & U_{1,j}^{(P)} & \cdots & U_{1,P+1}^{(P)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ U_{i,1}^{(P)} & \cdots & U_{i,j}^{(P)} & \cdots & U_{i,P+1}^{(P)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ U_{P+1,1}^{(P)} & \cdots & U_{P+1,j}^{(P)} & \cdots & U_{P+1,P+1}^{(P)} \end{bmatrix} \begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_i) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix},$$

where the block elements for $i = 1, \dots, P+1, j = 1, \dots, P$ are $U_{i,j}^{(P)} = v_j Z(s_i + T, s_j + \omega) B(s_j + \omega)$. The block elements in the last column of the matrix are given by $U_{i,P+1}^{(P)} = Z(s_i + T, 0) + v_{P+1} Z(s_i + T, \omega) B(\omega)$ for $i = 1, \dots, P+1$.

Equation (91) is based on the assumption of an exact representation of the fundamental solution $Z(t, \alpha)$ for $0 \leq \alpha \leq t \leq T$. However, we can in general only work with an approximate form for (91). Denote by $\widehat{Z}_N(t, \alpha)$ the collocated approximate fundamental solution for N Chebyshev points of $Z(t, \alpha)$ such that

$$(92) \quad \widehat{Z}_N(t, \alpha) \rightarrow Z(t, \alpha)$$

uniformly as $N \rightarrow \infty$ according to Theorem 3.1 for $0 \leq \alpha \leq t \leq T$. From Section 3.3 the matrix $\widehat{Z}_N(t, \alpha)$ is the concatenation of q matrices $\widehat{Z}_{iN}(t, \alpha)$ for $i = 1, 2, \dots, q$. Each of these i matrices $\widehat{Z}_{iN}(t, \alpha)$ can be written in the form

$$(93) \quad \widehat{Z}_{iN}(t, \alpha) = \sum_{k=0}^N W_{Nk}^{(i)} l_k \left(\frac{2}{\omega} t - \frac{2\alpha + (2i-1)\omega}{\omega} \right).$$

where $W_{Nk}^{(i)}$ is a matrix of collocation coefficients. Equation (90) now becomes

$$(94) \quad \begin{aligned} \left(U^{(PN)} \phi \right) (s_i) &= \sum_{j=1}^P v_j \widehat{Z}_N(s_i + T, s_j + \omega) B(s_j + \omega) \phi(s_j) \\ &+ \left(\widehat{Z}_N(s_i + T, 0) + v_{P+1} \widehat{Z}_N(s_i + T, \omega) B(\omega) \right) \phi(s_{P+1}). \end{aligned}$$

The matrix in (91) should now be written as

$$(95) \quad \begin{pmatrix} (U^{(PN)} \phi)(s_1) \\ \vdots \\ (U^{(PN)} \phi)(s_i) \\ \vdots \\ (U^{(PN)} \phi)(s_{P+1}) \end{pmatrix} = \begin{bmatrix} U_{1,1}^{(PN)} & \cdots & U_{1,j}^{(PN)} & \cdots & U_{1,P+1}^{(PN)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ U_{i,1}^{(PN)} & \cdots & U_{i,j}^{(PN)} & \cdots & U_{i,P+1}^{(PN)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ U_{P+1,1}^{(PN)} & \cdots & U_{P+1,j}^{(PN)} & \cdots & U_{P+1,P+1}^{(PN)} \end{bmatrix} \begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_i) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix},$$

where $U_{i,j}^{(PN)} = v_j \widehat{Z}_N(s_i + T, s_j + \omega) B(s_j + \omega)$ for $i = 1, \dots, P+1, j = 1, \dots, P$, and $U_{i,P+1}^{(PN)} = \widehat{Z}_N(s_i + T, 0) + v_{P+1} \widehat{Z}_N(s_i + T, \omega) B(\omega)$ for $i = 1, \dots, P+1$.

We note that it is not necessary to compute $\widehat{Z}_N(s_i + T, s_j + \omega)$ for every combination of (i, j) . The elements $U_{i,j}^{(PN)}$ in each column j above $U_{P+1,j}^{(PN)}$ contain the matrix factor $\widehat{Z}_N(s_i + T, s_j + \omega)$. Since $0 \leq T - \omega \leq s_i + T \leq T$ and $s_{P+1} = 0$, we need only develop the collocation polynomials for $\widehat{Z}_N(t, s_j + \omega)$ for $j = 1, \dots, P+1$ using (93) for all $t \in [s_j + \omega, T]$. Then each $\widehat{Z}_N(s_i + T, s_j + \omega)$ in the j -th column is simply an evaluation of the collocation polynomial developed for $\widehat{Z}_N(t, s_j + \omega)$ at the point $s_i + T$ for $i = i, \dots, P+1$. That is,

$$(96) \quad \widehat{Z}_N(s_i + T, s_j + \omega) = \sum_{k=0}^N W_{Nk}^{(i)} l_k \left(\frac{2}{\omega} (s_i + T) - \frac{2(s_j + \omega) + (2i-1)\omega}{\omega} \right).$$

This means we only need to develop the fundamental solution collocation representation for each element in row $P+1$, which involves $P+1$ computations of the approximate fundamental matrices rather than $(P+1)^2$ computations for the full matrix. All other rows involve interpolated values of the computed collocation polynomials at specific points in $[0, T]$.

From (95) the relevant eigenvalue problem becomes

$$(97) \quad \begin{bmatrix} U_{1,1}^{(PN)} & \cdots & U_{1,j}^{(PN)} & \cdots & U_{1,P+1}^{(PN)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ U_{i,1}^{(PN)} & \cdots & U_{i,j}^{(PN)} & \cdots & U_{i,P+1}^{(PN)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ U_{P+1,1}^{(PN)} & \cdots & U_{P+1,j}^{(PN)} & \cdots & U_{P+1,P+1}^{(PN)} \end{bmatrix} \begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_i) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix} = \lambda \begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_i) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix}.$$

The Matlab program *eig* can then be used to compute the eigenvalues and eigenvectors of (97). For some related discussion of the convergence of eigenvalues of finite matrix representations to operator eigenvalues see Baker [1].

5. Example

In this example we will consider the class of autonomous delay differential equations of the form

$$(98) \quad \ddot{x} + x = X(x(t-h), \dot{x}(t-h)),$$

where $x, X \in R, X(0, 0) = 0$. We assume that X is sufficiently differentiable. It is known that the solutions exist and are unique, if continuous initial condition functions are specified on the delay interval $[-h, 0]$ (Hale [18]). In order to simplify the notation, we will normalize the delay to unity. This can be done by

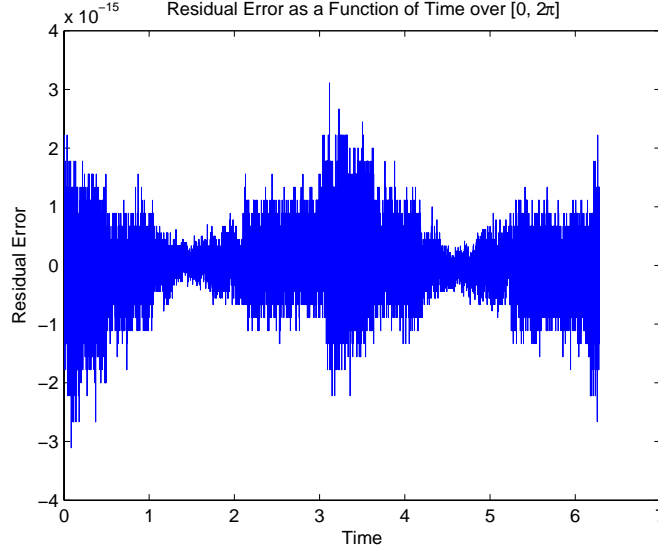


FIGURE 1. Residual Error of Approximate Solution for the Van der Pol Equation.

substituting th for t . Furthermore, since the period is unknown in (98) we can introduce a normalized period of $T = 2\pi$ by replacing t by t/ω where ω is an unknown frequency. Then we can put (98) into the form

$$(99) \quad \omega^2 \ddot{x} + x = X(x(t - \omega), \dot{x}(t - \omega)).$$

For this example we will look at the Van der Pol equation with

$$(100) \quad X(x(t - \omega), \dot{x}(t - \omega)) = \omega\lambda(1 - x(t - \omega)^2)\dot{x}(t - \omega).$$

A fast algorithm for constructing an approximate solution of (99) with (100) of the form

$$(101) \quad \tilde{x}(t) = a_0 + \sum_{n=1}^m [a_{2n} \cos nt + a_{2n-1} \sin nt]$$

along with an approximate frequency $a_{2m+1} = \tilde{\omega}$ has been given by Gilsinn [14].

To compare with an approximation result obtained in Stokes [36], we take $\lambda = 0.1$. By using Galerkin's method described in Section 5 the following approximate solution was obtained

$$(102) \quad \begin{aligned} \hat{x}(t) &= 2.0185 \cos(t) \\ &\quad + 2.5771 \times 10^{-3} \sin(2t) + 2.5655 \times 10^{-2} \cos(2t) \\ &\quad + 1.0667 \times 10^{-4} \sin(3t) - 5.2531 \times 10^{-4} \cos(3t) \\ &\quad - 7.1780 \times 10^{-6} \sin(4t) - 2.2043 \times 10^{-6} \cos(4t), \\ \hat{\omega} &= 1.0012. \end{aligned}$$

where we have displayed only the first few harmonics. This solution was estimated based on 11 harmonics, 40,000 sampled points over $[0, 2\pi]$, and 100 Chebyshev extreme points (24). For the quadrature steps in Section 4 P was taken as 200. This gave a mesh width of about $1/200$ on $[-\hat{\omega}, 0]$.

The residual was estimated by substituting $(\hat{\omega}, \hat{x})$ from equation (102) into equations (99) and (100) and finding the maximum of the absolute values of the residuals obtained in the interval $[0, 2\pi]$. The result was $r = 3.1086 \times 10^{-15}$. This residual is significantly better than the one given in Stokes [36]. The distribution of the residuals for the current case is shown in Figure 1. The phase plot of the approximate solution is shown in Figure 2.

The variational equation about the approximate periodic solution is

$$(103) \quad \dot{z} = \tilde{A}(t)z(t) + \tilde{B}(t)z(t - \tilde{\omega}),$$

where

$$(104) \quad \begin{aligned} \tilde{A}(t) &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{\tilde{\omega}^2} & 0 \end{bmatrix}, \\ \tilde{B}(t) &= \begin{bmatrix} 0 & 0 \\ -\frac{2\lambda}{\tilde{\omega}} \tilde{x}(t - \tilde{\omega}) \dot{\tilde{x}}(t - \tilde{\omega}) & \frac{\lambda}{\tilde{\omega}} (1 - \tilde{x}(t - \tilde{\omega})^2) \end{bmatrix}, \end{aligned}$$

where $\tilde{\omega}$ is the computed approximate frequency.

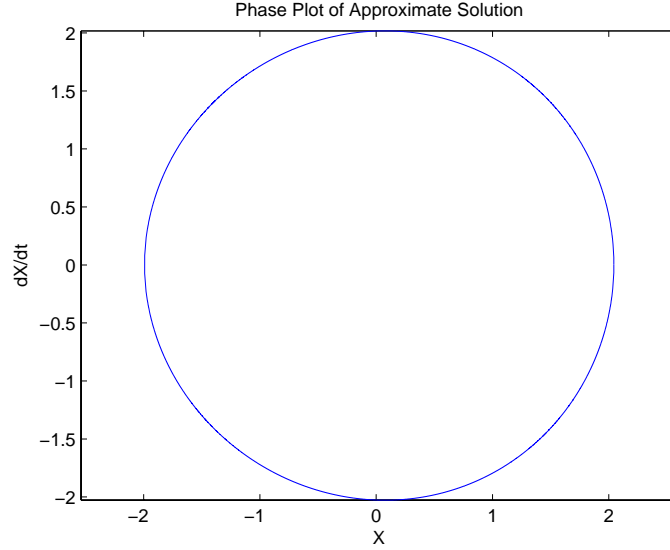


FIGURE 2. Phase Plot of Approximate Solution for the Van der Pol Equation.

We can now follow the development of the fundamental solution for (99) with the right hand side given by (100), as described in Section 3. At the first step we select the smallest positive integer q such that $2\pi \leq q\tilde{\omega}$. We then map each interval $[(i-1)\tilde{\omega}, i\tilde{\omega}]$ to $[-1, 1]$ for $i = 1, 2, \dots, q$ by

$$(105) \quad \begin{aligned} \eta &= \frac{2}{\tilde{\omega}}t - \frac{(2i-1)\tilde{\omega}}{\tilde{\omega}}, \\ t &= \frac{\tilde{\omega}}{2}\eta + \frac{(2i-1)\tilde{\omega}}{2}. \end{aligned}$$

We use the Matlab script *cheb.m* from Trefethen [39] to produce $N+1$ Chebyshev points, η_j , $j = 0, \dots, N$, as well as the spectral derivative matrix D (31). We then modify D so that the last row is all zeros except a one in position $D(N+1, N+1)$. We can then form $\hat{D} = D \otimes I_2$. In the case of the Van der Pol equation, the matrix \hat{A}_i is constant for each $i = 1, 2, \dots, q$, and is straightforward to construct. The matrix \hat{B}_i , for $i = 1, 2, \dots, q$, is not difficult to construct. For each η_j , $j = 0, \dots, N$ compute

$$(106) \quad t_j = \frac{\tilde{\omega}}{2}\eta_j + \frac{(2i-1)\tilde{\omega}}{2}.$$

The elements of \hat{B}_i , for $i = 1, 2, \dots, q$, in (41) are then formed by using (104) where we set $t = t_j$, $j = 0, \dots, N$. The matrix M_i in (44) can then be formed for each $i = 2, \dots, q$. For the case of $i = 1$ the matrix M_1 is formed as

$$(107) \quad M_1 = \left(\hat{D} - \frac{h}{2}\hat{A}_1 \right)^{-1}.$$

There are only two initial conditions to consider: $w_0(N+1) = 1, w_0(2(N+1)) = 0$ and $w_0(N+1) = 0, w_0(2(N+1)) = 1$. The weights at the first step are then given by (49). The final fundamental matrix is formed by following the steps from (50) to (53).

Figure 1 shows the distribution of the residuals. Figure 2 shows the approximate solution to the Van der Pol equation developed by the Galerkin procedure. Figure 3 shows the four components of the fundamental solution evaluated over $[0, 2\pi]$ by the collocation.

Figure 4 shows the results of plotting the first twenty eigenvalues of the discretized monodromy operator. The figure shows two eigenvalues symmetric about the real axis near the boundary of the unit circle. The other eighteen eigenvalues in both cases are so near the origin that they are all identified by an x at the origin. The result is consistent with the fact that the approximate periodic solution is not exact so that none of the eigenvalues of the variational equation with respect to the approximate solution need be exactly unity. However, they are sufficiently close to a magnitude of unity which most likely indicates a sufficiently good approximate solution in both cases. Also, the fact that all of the other eigenvalues are so near the origin is consistent with the fact that the monodromy operator and its discretizations are compact operators.

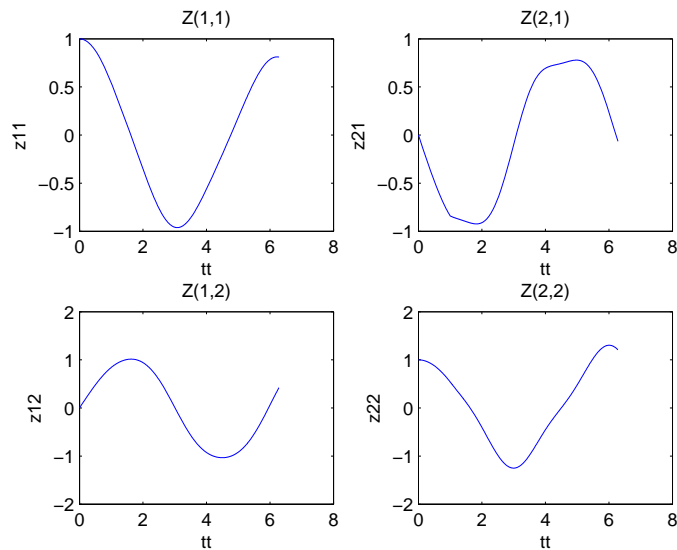


FIGURE 3. Fundamental Matrix for the Variational Equation relative to the Approximate Solution for the Van der Pol Equation.

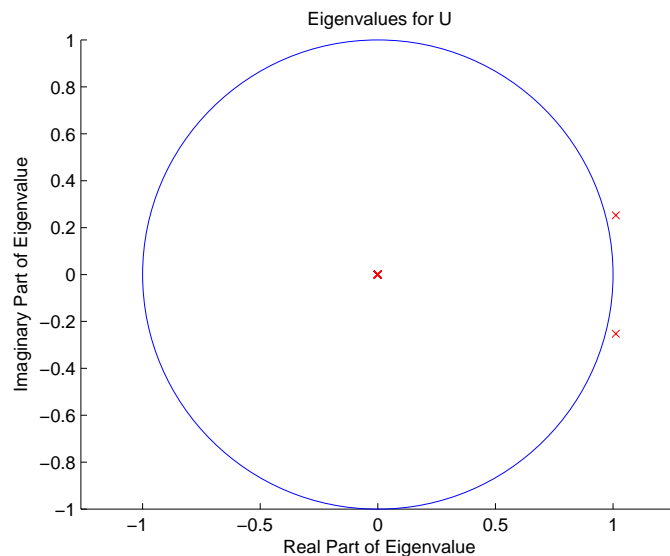


FIGURE 4. Eigenvalues for the Monodromy Operator

6. Disclaimer

Certain trade names and company products are mentioned in the text or identified in an illustration in order to adequately specify the experimental procedure and equipment used. In no case does such an identification imply recommendation or endorsement by the National Institute of Standards and Technology, nor does it imply that the products are necessarily the best available for the purpose.

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